2008

MATHEMATICS

First Paper

Unit - I

Time : Two hours Full Marks : 45

The figures in the margin indicate full marks.

Answer all from section A, three from section B

and one from section C.

Notations have their usual meanings.

Section - A

1. Show that every group of order 45 has a normal subgroup of order 9. 3

2. Let G be a finite commutative group. Prove that the number of solutions of $x^n = e$ in G where $n > 0$ and n divides $|G|$, is a multiple of n. 3

3. Show that no group of order 40 is simple. 3

Section - B

4. Prove that every homomorphic image of a solvable group is solvable. 6

5. Show that a group of order 15 is cyclic. 6

6. Let G be a group of order pqr where p, q, r are primes and $p > q > r$. Then show that G is solvable.

[Turn over

2008

MATHEMATICS

First Paper

Unit - II

Time : Two hours Full Marks : 45

The figures in the margin indicate full marks.

Section - A

Answer all the questions :

1. Either prove or disprove :

   A polynomial f of degree n over a commutative ring D with 1 has at most n distinct roots in some extension of D. 3

2. Show that a transcendental extension can not be finite. 3

3. Find the fixed field of Aut $\left(Q\left(\sqrt{5}\right)/Q\right)$. 3

Section - B

Answer any three questions :

4. If R is a PID and P is a nonzero proper ideal of R, then show that P is prime if and only if P is generated by a prime element. 6

5. Define characteristic of a ring. Show that the characteristic of an integral domain is either O or prime. 6

[Turn over

6. If K(u) is a simple transcendental extension of a field K then show that K(u) is isomorphic to the field of rational functions over K and any two simple transcendental extensions over K are isomorphic. 6

7. Let F be an extension of a field K of characteristic $p(\neq 0)$. Show that an element $\alpha \in F$, algebraic over K, is separable over K if and only if $K(\alpha)$ is separable over K. 6

8. If K is a field and $f \in K[x]$ has degree $n \geq 1$ then show that there exists a splitting field F of f with $[F:K] \leq n!$ 6

9. Show that a field of char. $p(\neq 0)$ is a perfect field if and only if every polynomial $(x^p - a)$ has a root in K, for all $a \in K$. 6

SECTION - C

Answer any one questions :

10. State and prove Eisenstein's irredoucibility criterion for polynomials over UFD. 18

11. If H is a finite set of automorphisms of a field F, then show that $|H| \leq [F:F_H]$, where F_H stands for the fixed field of H. Further show that the equality holds (i.e. $|H| = [F:F_H]$), if H is a group. 18

7. Let G be a nilpotent group of order m. If n > 0 and n/m then prove that G contains a subgroup of order n. 6

8. Show that any two composition series of a group G are equivalent. 6

9. Let G be a nilpotent group. Then prove that every subgroup of G is nilpotent. 6

SECTION - C

10. a) Show that any simple group of order 60 is isomorphic to A_5. 8

b) Prove that the derived subgroup G' of a group G is a normal subgroup of G. Also show that G/G' is commutative. 10

11. a) Define a subnormal series and a composition series in a group G. Show that a subnormal series in a group $G \neq \{e\}$ is a composition series if and only if it has no proper refinement. 2+2+6

b) Let G be a group. Then prove that G is solvable if and only if there is a positive integer m such that $G^{(m)} = \{e\}$. Hence show that the symmetric group s_n on n symbols is not solvable for $n \geq 5$. 5+3

8. Let $[X, s, \mu]$ be a measure space with $\mu(X) = 1$. If ψ is convex on (a, b) where $-\infty < a < b < \infty$, and f is a measurable function such that $a < f(x) < b$ for all x, prove that

$$\psi\left(\int f d\mu\right) \leq \int (\psi_0 f) d\mu.$$ 6

Section - C

9. Evaluate $\int_0^3 (x^2 + x + 1) d[x]$. 3

10. Show that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$ for any set B. 3

11. Let $f_n \in L^2(a, b), n = 1, 2, \cdots; f \in L^2(a, b)$ and $\| f_n - f \|_2 ; = 0$.

Show that $\int_a^b f^2 dx = \lim \int_a^b f_n{}^2 dx$. 3

2008

MATHEMATICS

Second Paper

Unit - I

Time : Two hours Full Marks : 45

The figures in the margin indicate full marks.

Answer one question from Section A, three from Section B and all from Section C.

Section - A

1. a) Let f be a bounded function on [a, b] and α be an increasing function on [a, b]. If P be a partition of [a, b] and P* be a refinement of P, prove that

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha).$$ 6

b) Let $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ for the partition

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}.$$

Show that $\sum_{i=1}^{n} | f(s_i) - f(t_i) | \Delta\alpha_i < \varepsilon$

where s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$. 6

[Turn over

2008

MATHEMATICS

SECOND PAPER

UNIT - II

Time : Two hours Full Marks : 45

The figures in the margin indicate full marks.

Answer ***one*** *question from Section A,* ***three*** *from Section B and* ***all*** *from Section C.*

SECTION - A

1. State and prove Stoke's theorem. 18

2. a) State and prove chain rule for vector valued functions of several variables. 12

   b) Find the relative extremum and saddle points for the function

   $$f(x,y)=x^3+x-4xy+2y^2$$ 6

SECTION - B

3. Suppose E is an open set in R^n and T is a C'-mapping of E into an open set V C R^m. Let ω and λ be k- and m-torms in V, respectively. Then prove that

   $$(\omega+\lambda)_T=\omega_T+\lambda_T \text{ if } k=m$$

   $$(\omega\wedge\lambda)_T=\omega_T\wedge\lambda_T$$ 6

[Turn over

4. Let $f: R^m \to R$ be differentiable on R^m. If R^m contains the line segment with end points a and a + h, then show that there is a piont $c = a + t_0 h$ with $0 < t_0 < 1$ of this line segment such that

$$f(a+h) - f(a) = Df(a)h$$ 6

5. Suppose ω is a k-form is an open set $E \subset R^m$, ϕ is a k-surface in E, with parameter domain $D \subset R^K$, and Δ is k-surface in R^K, with parameter domain D, defined by $\Delta(G) = u (u \in D)$. The show that $\int_\phi \omega = \int_\Delta \omega_\phi$ 6

6. State and prove Fubiri's theorem. 6

7. Prove that a function $f: R^n \to R^m$ is differentiable at $a \in R^n$ iff each component of f is differentiable at a. 6

8. a) Prove that the intersection of any collection of convex sets is a convex set. 3

   b) If $f: R^m \to R^n$ is a constant function, then show that f is differentiable. 3

SECTION - C

9. Define a differential form of order $k \geq 1$ in an open set $E \subset R^n$ and give an example of an 1-torm. 3

10. Define directional derivatives and partial derivatives of a function $f: R^n \to R$ at a point $a \in R^n$. 3

11. Let $Q \subset R^2$ be a rectangle and $f: Q \to R$ be a bounded function.

    Define $\underline{\int}_Q f$ and $\overline{\int}_Q f$. 3

2008

MATHEMATICS

Third Paper

Unit - I

Time : Two hours

Full Marks : 45

The figures in the margin indicate full marks.

Section - A

Answer any one from the following question

1. a) State Kuratowski's closure axioms. If X is a nonempty set and $C : P(X) \to P(X)$ is the Kuratowski closure operator on X, then show that there exists a unique topology τ on X such that for any $A \subseteq X$, C(A) coincides with τ-closure of A.

   (Here, P(X) stands for the power set of X) 2+7

   b) Show that every metric space is normal. 9

2. a) Show that the unit segment (0, 1) on the real line $\mathbb{R}$, the unit square $(0, 1) \times (0, 1)$ in the plane $\mathbb{R}^2$ and the unit cube $(0, 1) \times (0, 1) \times (0, 1)$ in the space $\mathbb{R}^3$ have the same cardinality. 8

   b) Show that two sets A, B have the same cardinality if there are injections $f : A \to B$ and $g : B \to A$. 6

c) Show that, $Cl(A \cup B) = Cl(A) \cup Cl(B)$ for any A, B, subsets of a topological space (X, τ).

Does the equality hold if 'union' in the above expression is replaced by 'intersection' ? – Justify.

(Here, Cl(A) stands for closure of A) 4

Section - B

Answer any three from the following questions :

3. Show that a topological space (X, τ) is T_2 if and only if the diagonal $\{(x,x) / x \in X\}$ is closed in $X \times X$. 6

4. Establish with examples that the product of two Lindelöff spaces need not be Lindelöff and subspace of a Lindelöff space need not be Lindelöff. 6

5. If f is a one-one map from a topological space (X, τ) onto a topological space (Y, σ), then show that f is a homeomorphism if and only if $f(\text{int} A) = \text{int} f(A)$, for all $A \subseteq X$. 6

6. Show that complete normality is a hereditary property. 6

7. Define closure of any subset A of a topological space (X, τ). Show that $Cl(A) = A \cup A'$ where Cl(A) denotes the closure of A and A' denotes the derived set of A. 6

8. Let X be an infinite set and τ be defined as follows :

$\tau = \{U \subseteq X / X \setminus U \text{ is finite or } \phi\} \cup \{\phi\}$.

Show that, (X, τ) is Lindelöff, T_1 but not T_2. 6

Section - C

*Answer **all** from the following :*

9. Show that the function $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{1 - x^2}$ is a homeomorphism. 3

10. Let A and B be two nonempty sets. Prove that card (B) $\leq$ card (A) if and only if there is a surjection from A onto B. 4

11. State Urysohn's lemma. 2

5. Prove that every closed subspace of a paracompact space is paracompact.

6. Prove that a compact Haurdorff space is normal.

7. Prove that if $\{F_n\}$ is a sequence of non-empty closed sets in a compact space with the property $F_n \supseteq F_{n+1} \forall n \in \mathbb{N}$ then $\bigcap_{n=1}^{\alpha} F_n \neq \alpha$.

8. Prove that the product space of a family of topological spaces $\{(X_\alpha, Y_\alpha); \alpha \in \wedge\}$ completely regular if and only if each (X_α, Y_α) in the family is completely regular.

9. If $\{C_\alpha \; ; \alpha \in \wedge\}$ is a family of connected sets in a topological space and $\bigcap_{\alpha \in \wedge} C_\alpha \neq \phi$ then show that $\bigcup_{\alpha \in \wedge} C_\alpha$ is connected.

Group - C

18×1=18

10. i) Show that a space is metrizable if and only if X is regular and has a basis that is countably locally finite. 9

ii) If $\{(X_\alpha, Y_\alpha); \alpha \in \wedge\}$ in an arbitrary family of compact topological spaces than show that the corresponding product space in also compact. 9

11. i) In a metrizable space X, show that the following are equivalent.

a) X is compact

b) X is sequentially compact

c) X is Freehet compact.

d) X is countably compact. 8

ii) Let $p: E \to B$ be a covering map ; let $p(e_0) = b_0$. If the map $F: I \times I \to B$ be continuous with $F(0, 0) = b_0$, then prove that there is a unique lifting of F to a continuous map $\tilde{F}: I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If further F is a path homotopy then show that $\tilde{F}$ is also a path homotopy. 10

PG-I(DE)/08(New)

2008

MATHEMATICS

FOURTH PAPER

UNIT - II

Time : Two hours Full Marks : 45

The figures in the margin indicate full marks.
*Answer **all** from Section A, **three** from Section B and **one** from Section C.*

Notations have their usual meanings.

SECTION - A

1. Find the conjugate harmonic function v(x, y) corresponding to $u(x,y) = x^2 - y^2 + x$ on the domain $|z| < \infty$. 3

2. Find the order of the polynomial function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad \text{where } a_n \neq 0.$$ 3

3. Define a mermorphic function with an example. State the function theoretic behaviour of the point at infinity in the case of meromorphic functions. 3

SECTION - B

4. Stating the mean value property prove that any harmonic function defined in a domain D has the mean value property. 6

[Turn over

5. Establish that

$$\frac{P^2 - r^2}{P^2 + r^2 - 2Pr\cos(\theta - \phi)} = 1 + 2\sum_{n=1}^{\alpha}\left(\frac{r}{P}\right)^n \cos n(\theta - \phi)$$

and $\frac{2Pr\sin(\theta - \phi)}{P^2 + r^2 - 2Pr\cos(\theta - \phi)} = 2\sum_{n=1}^{\alpha}\left(\frac{r}{P}\right)^n \sin n(\theta - \phi)$ in

$|z - z_0| < P$. 6

6. If an entire function is not a polynomial then show that it is non algebraic. 6

7. Represent $\sin \pi z$ in the form of canonical product. 6

8. Show that if p is a positive integer, then $|E(z,p)| \le b\exp$ $(a|z|p)$ for suitable a, b. 6

9. If f(z) is an entire function and $f(0) \ne 0$, then show that $f(z) = f(0)p(z)_e p(z)$ where p(z) is a product of weirstrass primary factors. 6

Section - C

10. State and prove the Dirichlet Problem for a disc related to harmonic functions. Hence show that the Dirichlet problem for a disc has a unique solution. 18

11. a) If $z_1, z_2, \ldots z_n$ be any sequence of numbers whose only limiting point is the point at infinity, then show that it is possible to construct an integral function which vanishes at each of these points z_n. 9

b) If f(z) is an entire function of order P, then show that for every $\varepsilon > 0$ the inequality $N(r) \le r^{P+\varepsilon}$ holds for all sufficiently large r where n(r) is the number of zeros of f(z) in $|z| \le r$. 9

2008

MATHEMATICS

FIFTH PAPER

UNIT - I

Time : Two hours | Full Marks : 45

The figures in the margin indicate full marks.

*Answer **all** from Section A, **three** from Section B*

*and **one** from Section C.*

Notations have their usual meanings.

SECTION - A

1. a) Define the following terms : $2\times4=8$

   i) Closure of a function,

   ii) Locally finite open covering,

   iii) C^∞-partition of unity,

   iv) paracompact space.

   b) State Urysohn's metrization theorem. 2

   c) What do you mean by a differentiable manifold ? Prove that S' is a differentiable manifold of dimension 1. 2+6

2. a) What do you mean by a Lie bracket ? 2

   b) Prove that for all $X, Y, Z \in \chi(M)$ and for all $f, g \in C^\infty(M)$.

[Turn over

i) $[X, Y](f+g) = [X, Y]f + [X, Y]g$;

ii) $[X, Y]fg = f[X, Y]g + g[X, Y]f$;

iii) $[X, Y+Z] = [X, Y] + [X, Z]$,

iv) $[X, X] = 0$

v) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

2+2+2+2+4

c) If $X, Y \in \chi(\mathbb{R}^3)$ are defined by

$$X = \frac{\partial}{\partial x^1}, Y = \frac{\partial}{\partial x^2} + e^{x^1}\frac{\partial}{\partial x^3}, \text{ compute } [X, Y].$$ 4

SECTION - B

3. What do you mean by an integral curve of a vector field ? Find the integral curve of X in $\mathbb{R}^2$ where

$$X = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}.$$ 2+4

4. What do you mean by an one parameter group of transformation?

Prove that $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\phi(t, p) = \phi_t(p)$,

precisely given by for $p = (x^1, x^2) \in \mathbb{R}^2$

$\phi_t(p) = (\overline{x}^1, \overline{x}^2) = (x^1 \cos t - x^2 \sin t, x^1 \sin t + x^2 \cos t)$

is a one parameter group of transformation. 2+4

5. If $X = \sum X^i \frac{\partial}{\partial x^i}$, $Y = \sum Y^i \frac{\partial}{\partial x^i}$; then find an expression for $[X, Y]$. Given X, Y are vector fields on $\mathbb{R}^n$. 6

6. What do you mean by a r-form ? Define the wedge product of two 1-forms. Prove that the wedge product of two 1-forms is skew-symmetric. 2+2+2

7. Define a Lie group. Show that $\mathbb{R}^n$ is a Lie group. 2+4

SECTION - C

Answer all questions

8. If X is a tangent vector on a manifold M, show that $X_C = 0$ where $C : M \to \mathbb{R}$ is a constant function. 3

9. Define a vector bundle with an example. 3

10. Compute $(9du^1 + 3du^2) \wedge (3du^1 + 2du^2)$. 3

PG-I(DE)/08(New)

2008

MATHEMATICS

FOURTH PAPER

UNIT - I

Time : Two hours Full Marks : 45

The figures in the margin indicate full marks.

*Answer **all** from Section A, **three** from Section B and **one** from Section C.*

Notations have their usual meanings.

SECTION - A

1. If a function f has an essential singularity at the point ' α ' then show that $\frac{1}{f}$ has also an essential singularity at ' α '. 3

2. Prove that a function which is analytic everywhere including the point at infinity is a constant. 3

3. Show that the limit point of poles of a function f is a non isolated essential singularity of f. 3

SECTION - B

4. Prove that all the roots of the equation $z^7 - 5z^3 + 12 = 0$ lie in $1 < |z| < 2$. State the related theorem. 6

5. Find the bilinear transformation which maps the points 1, i, –1 in the z-plane into the points 0, 1, α in the w-plane. Show that by means of this transformation the area of the circle $|z|=1$ is represented in the w-plane by the half plane above the real axis. 6

[Turn over

PG-I(DE)/08(New)

2008

MATHEMATICS

FIFTH PAPER

UNIT - II

Time : Two hours Full Marks : 45

The figures in the margin indicate full marks.

*Answer **one** from Section A, **three** from Section B and **all** from Section C.*

Notations Carry Their Usual Meanings.

SECTION - A

1. a) What do you mean submanifold of a manifold ? 2

   b) Let M^n be a submanifold of N^m, where both M and N are Riemannian manifolds. If $X, Y \in \chi(M)$ with $\bar{X}, \bar{Y}$ as their extensions on N, then prove that $[\bar{X}, \bar{Y}] = [X, Y]$. 4

   c) Deduce the equations of Codazzi and Ricci. 6+6

2. a) Define the following terms :

   i) torsion tensor, ii) curvature tensor,

   iii) Ricci tensor, iv) Curvature scalar.

   b) State and prove the fundamental theorem of Riemannian geometry.

[Turn over

Section - B

3. Prove that on an n-dimensional smooth manifold there always exists a Riemannian metric. 6

4. What do you mean by an affine connection on a manifold ? Cite an example. when an affine connection is said to be symmetric ? 2+2+2

5. Show that every Riemannian manifold of dimension 3 is of constant curvature. 6

6. For all $X, Y, Z \in \chi(M)$ prove that

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$$

where ∇ denotes the Riemannian connection. 6

7. Define 'sectional curvature'. State Schur's theorem. Using Schur's theorem what can we conclude about a Riemannian manifold of dimension ≥ 3 which is everywhere wandering ? 2+2+2

Section - C

8. Define curvature tensor and prove that it is skew-symmetric. 1+2

9. What do you mean by a complex structure ? Explain. 3

10. If for a Riemannian manifold (M^n, g), the torsion tensor T satisfies the relation

$$T(X,Y) = u(Y)X - u(X)Y \text{ for all } X, Y \in \chi(M)$$

where u is a 1-form, find an simplified expression for

$$g(T(X,Y),Z) + g(T(Y,Z),X) + g(T(Z,X),Y).$$ 3

6. Show that $\text{Sin}\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} an\left(z^n + \frac{1}{z^n}\right)$, where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \text{Sin}(2\cos\theta)\text{Cos}n\theta d\theta.$$ 6

7. State and prove Liouville's theorem. 6

8. If f is analytic within and on a closed contour C and $f(z) \neq 0$ on C then show that the number of zeros of f within C is given by

$$N = \frac{1}{2\pi}[\arg f(z)]_C.$$ 6

9. Find all the bilinear transformations of the half plane Im $(z) \geq 0$ into the unit circle $|w| \leq 1$. 6

Section - C

10. a) State and prove the necessary and sufficient condition for a function to be analytic at a point. 12

b) Show that $\left|\frac{z-p}{z-q}\right| = K(\neq 1)$ represents a circle with respect to which p and q are inverse points. 6

11. a) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent's series valid for $0 < |z+1| < 2$. 3

b) Evaluate $\int_0^{\infty} \frac{x^{a-1}}{1+x} dx$ by contour integration method and hence deduce that $\int_0^{\infty} \frac{x^{a-1}}{1-x} dx = \pi \text{Cot} a\pi$ where $0<a<1$. 15

PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—I (Unit—I)

Time : 2 hours | *Full Marks* : 45

The figures in the margin indicate full marks.

Answer **all** from Section—A, **three** from Section—B and **one** from Section—C.

Notations have their usual meanings.

SECTION—A

1. If H and K are subgroups of a group G, then prove that $N_K(H)$ is a subgroup of K. 3

2. Let H be a normal subgroup of a group G. If H and $G|H$ are both p-groups, then show that G is a p-group. 3

3. Prove that no group of order 10 is simple. 3

SECTION—B

4. Let G be a group of order 455. Show that G is cyclic. 6

5. Let G be a nilpotent group of order m. If $n > 0$ and $n|m$, then prove that G contains a subgroup of order n. 6

6. Let G be a group of order p^n where p is prime and $n \in z$, $n \geq 1$. Prove that any subgroup of G of order p^{n-1} is normal in G. 6

7. Show that every finite p-group is nilpotent. 6

8. Prove that a group of order 130 contains a non-trivial normal subgroup. 6

9. Show that there are only two groups of order 4 (up to isomorphism). 6

SECTION—C

10. *(a)* Establish Sylow's first theorem. 10

    (b) Show that a group of order 96 has a normal subgroup of order 16 or 32. 8

11. Let G be a group of order 231. Then—

    (a) show that a Sylow 2-subgroup of G is normal in G;

    (b) show that a Sylow 7-subgroup of G is normal in G;

    (c) show that G has a cyclic subgroup of order 77;

    (d) if H be a Sylow 2-subgroup of G, K be a Sylow 7-subgroup of G and L be a Sylow 3-subgroup of G, show that $G = HKL$;

    (e) show that $H \subseteq z(G)$. 18

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PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—I (Unit—II)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Unless otherwise stated, notations carry their usual meanings

SECTION—A

Answer *any* **three** questions.

1. State and prove Artin's theorem regarding extension of fields. 6

2. Define an Euclidean domain. Prove that every Euclidean domain is a Principal Ideal Domain. 2+4

3. Define a prime element and an irreducible element in a commutative ring R with 1. Cite an example of an element in $\mathbb{Z}_6$ which is prime but not irreducible. 1+1+4

4. Let R be a commutative ring with 1 and M is an ideal of R. Prove that M is a maximal ideal of R if and only if R / M is a field. 6

5. What do you mean by a field extension? Cite an example. Prove that every finite extension is algebraic. 1+1+4

6. Let A be an $n \times n$ matrix over a field F. What do you mean by the Smith normal form of A? Describe briefly how you will convert an $n \times n$ matrix to rational canonical form. 2+4

SECTION—B

Answer *any* **one** question.

7. (a) State and prove Hilbert basis theorem. 9

   (b) Let R be a commutative ring with 1. Also let f, g are polynomials in $R[x]$, where the leading coefficient of g is a unit in R. Prove that there exists unique polynomials q, r in $R[x]$ such that $f = gq + r$, where $r = 0$ or $\deg(r) < \deg(g)$. 9

8. (a) What do you mean by a ring embedding? Prove that any integral domain can be embedded in a field. 2+7

   (b) Let F / K be a field extension and $u \in F$. Prove that u is algebraic over K if and only if u is root of some unique irreducible monic polynomial $p(x) \in K[x]$. 9

SECTION—C

Answer **all** questions.

9. Find a basis of $Q(\sqrt{2}, \sqrt{3})$ over Q. 3

10. What do you mean by splitting field of a polynomial $f(x) \in K[x]$, where K is a field? Describe with a suitable example. 3

11. Prove that any automorphism over a field fixes its prime subfield. 3

★ ★ ★

8. Let $\{f_n\}$ be a sequence of integrable functions such that

$$\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$$

Prove that $\sum_{n=1}^{\infty} f_n$ converges a.e., its sum, f, is integrable and

$$\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$ 6

Section—C

9. The functions $f(x)$ and $\alpha(x)$ defined on $[-1, 1]$ are given by

$$f(x) = 0 \text{ for } -1 \le x \le 0$$
$$= 1 \text{ for } 0 < x \le 1$$
$$g(x) = 0 \text{ for } -1 \le x < 0$$
$$= 1 \text{ for } 0 \le x \le 1$$

Show that $f \in R(\alpha)$ on $[-1, 1]$ and $\int_{-1}^{1} f \, d\alpha = 0$. 3

10. Prove that every interval is both an F_σ and a G_δ. 3

11. Show that if a sequence of measurable functions converges in measure, then the limit function is unique a.e. 3

★ ★ ★

PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—II (Unit—I)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Answer **one** question from Section—A, **three** from Section—B and **all** from Section—C.

Symbols carry their usual meanings.

Section—A

1. (a) When is a bounded function f on $[a, b]$ said to be Riemann-Stieltjes integrable w.r.t. an increasing function α? Prove that $f \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$ 2+4

(b) Show that if $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$, then $f \in R(\alpha_1 + \alpha_2)$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d(\alpha_1) + \int_a^b f d(\alpha_2)$$ 5

(c) Let—

(i) ϕ is a strictly increasing and continuous function that maps the interval $[A, B]$ onto $[a, b]$;

(ii) α is monotonically increasing on $[a, b]$ and $f \in R(\alpha)$ on $[a, b]$;

(iii) the functions β and g are defined by $\beta(y) = \alpha(\phi(y))$, $g(y) = f(\phi(y))$.

Prove that $g \in R(\beta)$ and

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha$$ 7

2. *(a)* What do you mean by the Lebesgue measurability of a set of real numbers? Prove that every interval is measurable. 1+5

(b) Define two sequences $\{f_n\}$ and $\{g_n\}$ as follows :

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \text{ if } x \in R, \quad n = 1, 2, \ldots,$$

$$g_n(x) = \begin{cases} \dfrac{1}{n} & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ b + \dfrac{1}{n} & \text{if } x \text{ is rational, say } x = \dfrac{a}{b},\ b > 0 \end{cases}$$

Let $h_n(x) = f_n(x) g_n(x)$. Prove that both $\{f_n\}$ and $\{g_n\}$ converge uniformly on every bounded interval but $\{h_n\}$ does not converge uniformly on any bounded interval. 5

(c) Let $\{f_n\}$ be a sequence of non-negative measurable functions. Prove that

$$\liminf \int f_n\, dx \ge \int \liminf f_n\, dx$$ 7

SECTION—B

3. Let $f_1, f_2 \in BV([a, b])$ and $c_1, c_2 \in \mathbb{R}$. Prove that $c_1 f_1 + c_2 f_2 \in BV([a, b])$ and

$$V_a^b(c_1 f_1 + c_2 f_2) \le |c_1| V_a^b(f_1) + |c_2| V_a^b(f_2)$$ 6

4. Let f and g be real-valued measurable functions defined on the same measurable domain. Prove that $f + g$ is measurable. 6

5. If f, g are vector-valued functions from $[a, b]$ to $\mathbb{R}^k$ with $f \in R(\alpha)$, $g \in R(\alpha)$, then prove that $f + g \in R(\alpha)$ and

$$\int_a^b (f + g)\, d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha$$ 6

6. Let $f \in L^p(a, b)$ with a and b finite and $p \ge 1$, and let $\varepsilon > 0$. Show that there exists a step function h such that

$$\int_a^b |f - h|^p\, dx < \varepsilon$$ 6

7. Let $p \ge 1$ and let $f, g \in L^p(\mu)$. Prove that

$$\left(\int |f + g|^p\, d\mu\right)^{1/p} \le \left(\int |f|^p\, d\mu\right)^{1/p} + \left(\int |g|^p\, d\mu\right)^{1/p}$$ 6

PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—II (Unit—II)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Answer **one** question from Section—A, **three** from Section—B and **all** from Section—C.

SECTION—A

1. If $f : R^m \to R^n$ be a function such that the partial derivatives

$$D_j f_i(x)\ (j = 1, 2, ..., m;\ i = 1, 2, ..., n)$$

at x are continuous, then show that f is differentiable at x. Is the converse true? Give reasons for your answer. 18

2. *(a)* Let Q be a rectangle in R^n and $f : Q \to R$ be a bounded function. Let P and P'' be any two partitions of Q and P' be a refinement of P. Prove that

(i) $L[f, P] \le L[f, P']$

(ii) $U[f, P] \ge U[f, P']$

(iii) $L[f, P] \le U[f, P'']$

(iv) $\underline{\int}_Q f \le \overline{\int}_Q f$ 3+3+3+3

(b) State and prove a necessary and sufficient condition for Riemann integrability. 6

SECTION—B

3. State and prove implicit function theorem. 6

4. If $f : R^m \to R$ be a function of class C^2, then prove that for each point $x \in R^m$, $D_k D_j f(x) = D_j D_k f(x)$. 6

5. Let $g : R^2 \to R^2$ be defined by

$$g(x, y) = (2ye^{2x}, xe^y)$$

Show that g maps a neighbourhood of (0, 1) injectively onto a neighbourhood of (2, 0). If $f : R^m \to R^n$ is differentiable at $x \in R^n$, then show that f is continuous there. 6

6. If X is a closed and bounded subspace of R^n, then prove that X is compact. 6

7. If $f : R^n \to R^m$ is differentiable at x, then prove that the partial derivatives $(D_j f_i)(x)$ exist and

$$f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i \quad (1 \le j \le n)$$

where $\{e_1, e_2, ..., e_n\}$ and $\{u_1, u_2, ..., u_m\}$ are the standard bases of R^n and R^m respectively. 6

8. Find the maximum and minimum of the rectangular solid in the first octant $\{x \ge 0, y \ge 0, z \ge 0\}$ with one vertex at the origin and the opposite vertices lying on the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, c are positive constants. 6

SECTION—C

9. Define wedge product of two 1-forms. 3

10. Let $f : R^3 \to R^2$ be such that

$$Df(0) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

and $f(0) = (2, 3)$ and let $g : R^2 \to R^2$ be such that

$$Dg(2, 3) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

Find $D(gf)(0)$. 3

11. Prove by example that Lagranges mean value theorem does not hold for a vector-valued function. 3

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PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—III (Unit—I)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

SECTION—A

Answer *any* **one** question.

1. *(a)* Prove that a topological space (X, T) is normal iff for every pair of disjoint closed subsets F_1 and F_2 of X and closed interval $[0, 1]$ of R, there exists a continuous mapping $f: X \to [0, 1]$ such that $f(F_1) = \{0\}$ and $f(F_2) = \{1\}$.

(b) Give an example of a topological space (X, T) whose T-open set and T-closed set are same. 15+3

2. *(a)* Show that each singleton subset of a Hausdorff space is closed.

(b) Let (X, T) and (X, T^*) are two topological spaces such that T^* is finer than T. If (X, T) is a T_2-space, then show that (X, T^*) is also a T_2-space.

(c) Show that a topological space (X, T) is regular iff each $x \in X$ $\exists$ a neighbourhood of x and a T-open set G containing

$$x : x \in G \subset \overline{G} \subset N$$ 6+5+7

SECTION—B

Answer *any* **three** questions.

3. (a) Let $X = \{a, b, c, d\}$, $T = [\phi, X, \{a\}, \{b\}, \{c\}, \{ab\}, \{abc\}, \{bc\}, \{ca\}, \{abd\}, \{abcd\}]$. Find a base for (X, T).

   (b) Define a local base for a topological space. Give an example. Show that (R, D) is a first countable space. 6

4. Show that $[0, 1]$ is non-enumerable. 6

5. Prove that every second countable space is a Lindelöf space. Is the converse true? 6

6. Show that Tychonoff space is a T_3-space. 6

7. Define a normal space and completely normal space. Give one example of each. Let (X, T) is a Hausdorff space. Then prove that every convergent sequence has a unique limit. 6

8. If (X, T) be a topological space and A, B are two subsets of X, then prove that—

   (i) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$

   (ii) $\overline{(A \cap B)} \subset \overline{A} \cap \overline{B}$

   Give an example to show that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$. 6

SECTION—C

9. (a) Show that the space of real numbers with usual topology is Hausdorff.

   (b) State Schroder-Bernstein theorem.

   (c) What is the cardinal number of $N \times N$? (N is the set of natural numbers.) 9

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PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—III (Unit—II)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Answer **all** from Group—A, **three** from Group—B and **one** from Group—C.

Group—A

1. Prove the Urysohn's lemma on metrizable spaces. 3

2. Show by an example that the product of two paracompact spaces need not be paracompact. 3

3. If X and Y be any two compact subsets of $\mathbb{R}$, then make a conclusion about the compactness of $X+Y$. 3

Group—B

4. Show that every compact subset of a Hausdorff space is closed. 6

5. If X and Y are two connected spaces, then show that $X \times Y$ is also connected. 6

6. Let $f: X \to Y$ and Y be a compact Hausdorff space. Then show that f is continuous if and only if the graph of f, $G = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$. 6

7. Let F be a locally finite collection of subsets of a topological space X. Then prove that—

(a) the collection $C = \{\overline{A}\}_{A \in F}$ of the closures of the elements of F is locally finite.

(b) $\overline{\bigcup_{A \in F} A} = \bigcup_{A \in F} \overline{A}$ 3+3

8. Show that the fundamental group of S^1 is isomorphic to the additive group of integers. 6

9. Let $S: D \to X$ be a net in a topological space and let $x \in X$. Then show that x is a cluster point of S if and only if there exists a subnet of S converging to x in X. 6

GROUP—C

10. (a) Show that every metrizable space is paracompact. 10

(b) Show that if $\prod_{\alpha \in \Lambda} X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α. 8

11. (a) Show that a sequentially compact metrizable space is totally bounded and in this space every open cover has a Lebesgue number. 6+6

(b) Show that every countably compact space is Frechet compact. 6

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PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—IV (Unit—I)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Answer **all** from Section—A, **three** from Section—B and **one** from Section—C.

Notations have their usual meanings.

SECTION—A

1. Show that a function $f(z)$, which is regular everywhere except at infinity where it has a pole of order n, is a polynomial of degree n. 3

2. Find the residue of $\dfrac{z^2}{z^2+a^2}$ at $z = ia$. 3

3. Evaluate $\int_C |dz|$. 3

SECTION—B

4. State and prove Cauchy's integral formula for a simply connected domain. 6

5. Prove that the zeros of an analytic function are isolated points. 6

6. Establish maximum modulus theorem. 6

7. Prove that one root of the equation $z^4 + z^3 + 1 = 0$ lies in the first quadrant. 6

8. Use the method of contour integration to prove that $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)^3} = \frac{\pi}{8a^3}$. 6

9. Show that both the transformations $w = \frac{1-z}{1+z}$ and $w = \frac{z-1}{z+1}$ transform $|w| \leq 1$ into the half-plane $\text{Re}(z) \geq 0$. 6

SECTION—C

10. (a) For the function $f(z) = \frac{2z^3 + 1}{z^2 + z}$, find—

(i) a Taylor series valid in the neighbourhood of the point $z = i$;

(ii) a Laurent series valid within the annulus of which centre is origin. 10

(b) Establish Cauchy's residue theorem. 8

11. (a) Show that there cannot be more than one continuation of the same analytic function with the same domain. 7

(b) If two functions, which are analytic in a domain D, coincide in a part of D, then prove that they coincide in the whole of domain D. 4

(c) Evaluate : 7

$$\int_0^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)(x^2 + b^2)} dx \quad (a > 0,\ b > 0)$$

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PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—IV (Unit—II)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Answer **all** from Section—A, **three** from Section—B and **one** from Section—C.

The notations have their usual meanings.

Section—A

1. Prove that a continuous function u in a domain D with mean value property is necessarily harmonic in D. 3

2. If ρ_1 and ρ_2 be the orders of the entire functions $f_1(z)$ and $f_2(z)$ respectively, then show that the order of their product $f_1 f_2 \le \max(\rho_1, \rho_2)$. 3

3. Show that the exponent of convergence of zeros of $\cos z$ is 1. 3

SECTION—B

4. State and prove Hadamard's three-circle theorem. 6

5. Using Mittag-Leffler theorem, show that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} \quad 6$$

6. Establish Poisson's integral theorem related to harmonic functions in a circular domain. 6

7. State and prove Riemann mapping theorem. 6

8. If $f(z)$ is an entire function which never vanishes, then prove that there exists an entire function $g(z)$ such that $f(z) = e^{g(z)}$ for all $z \in ¢$. 6

9. Establish Hurwitz's theorem for a sequence of analytic functions. 6

SECTION—C

10. (a) Prove that a family F of analytic functions on D is normal iff the functions in F are uniformly bounded on every compact subset in D. 7

(b) Show that the order of a canonical product is equal to the exponent of convergence of its zeros. 7

(c) Let $f(z)$ be analytic in $D : |z| \le R$. Then prove that for each $0 < r < R$

$$M(r) \le \frac{2r}{R-r} A(r) + \frac{R+r}{R-r} |f(0)| \quad 4$$

11. (a) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of finite order ρ. Then prove that

$$\limsup_{n \to \infty} \frac{\log n}{\log |a_n|^{-1/n}} = \rho \quad 7$$

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be entire with finite (nonzero) order ρ and of the type σ. Then show that $\frac{1}{e\rho} \limsup_{n \to \infty} n |a_n|^{\rho/n} = \sigma$. 7

(c) Establish Harnack's inequality related to a harmonic function. 4

★ ★ ★

10. Given $X, Y \in \chi(\mathbb{R}^2)$ with

$$X = x^2 \frac{\partial}{\partial x}, \ Y = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$$

find $[X, Y]$. 3

11. What do you mean by Principal Fibre Bundle? 3

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PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—V (Unit—I)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Unless otherwise stated, notations carry their usual meanings.

SECTION—A

Answer *any* **three** questions.

1. Prove that the real projective space $\mathbb{R}P^n$ is a differentiable manifold. 6

2. What do you mean by a Lie algebra? Show that the collection $M(n, \mathbb{R})$ of all $n \times n$ real matrices can be given a Lie algebra structure. 6

3. If

$$\omega \in \Lambda^k(V^*), \ \eta \in \Lambda^l(V^*), \ \pi \in \Lambda^m(V^*)$$

where V is an n-dimensional real vector space, prove that

$$\omega \wedge (\eta \wedge \pi) = (\omega \wedge \eta) \wedge \pi$$ 6

4. Let G be a Lie group and $a, b \in G$. What do you mean by left translation L_a in G? Prove or disprove the following statements : 2+2+2

(a) $L_a L_b = L_{ab}$

(b) $L_{a^{-1}} = (L_a)^{-1}$

5. Given

$$X = x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2} \in \chi(\mathbb{R}^2)$$

Find the equation for integral curve for X. 6

6. What do you mean by f-related vector fields? Let f be a smooth map between two smooth manifolds M and N. If X_i and Y_i $(i = 1, 2)$ are f-related vector fields on M and N respectively, then prove that $[X_1, X_2]$ and $[Y_1, Y_2]$ are also f-related. 1+5

SECTION—B

Answer *any* **one** question.

7. (a) Define a Lie group. Show that $GL(n, \mathbb{R})$ is a Lie group. Find its dimension. 2+6+1

(b) What do you mean by a Lie bracket? For $X, Y \in \chi(M)$; $f, g \in C^\infty(M)$, find a simplified expression for $[fX, gY]$. Also prove that

$$[Y, X] = -[X, Y]$$ 1+6+2

8. (a) Let M be a differentiable manifold and $p \in M$. Also let (u^i) be a local coordinate system $\{i = 1, 2, ..., n\}$ in some neighbourhood of p. Define a tangent vector X_p as a directional derivative operator. Also prove that $T_p M$ is a vector space with

$$\left\{ \frac{\partial}{\partial u^i}\Big|_p : i = 1, 2, ..., n \right\}$$

as a basis. 3+6

(b) Let $X, Y \in \chi(M)$. If X generates a local one parameter group of transformations ϕ_t, then prove that

$$[X, Y] = \lim_{t \to 0} \frac{1}{t}\{Y - (\phi_t)_* Y\}$$ 9

SECTION—C

Answer **all** questions.

9. Compute : 3

$$(27du^1 + 13du^2) \wedge (du^1 + 3du^2)$$

PG Part–I (Under DE Mode) Exam., 2009

MATHEMATICS

Paper—V (Unit—II)

Time : 2 hours | *Full Marks* : 45

The figures in the margin indicate full marks.

Unless otherwise stated, notations carry their usual meanings.

SECTION—A

Answer *any* **three** questions.

1. What do you mean by a Tachibana manifold? Prove that a Tachibana manifold is a Kahler manifold. 2+4

2. State and prove first Bianchi identity. 6

3. Let M be a Riemannian manifold and $p \in M$. Define sectional curvature at the point $p \in M$. When the manifold is said to be wandering at the point $p \in M$? State Schur's theorem. 2+2+2

4. State and prove Ricci identity for a tensor field of type (0, 1). 6

5. Define a complex structure on a finite dimensional real vector space. Show that a real vector space with a complex structure is even dimensional. 2+4

6. What do you mean by holomorphic sectional curvature of a complex manifold? Prove that a Kahler manifold of constant holomorphic sectional curvature is an Einstein manifold. 6

SECTION—B

Answer *any* **one** question.

7. *(a)* Deduce equations of Gauss and Codazzi. 9

(b) Define an affine connection on a smooth manifold. Define torsion tensor of an affine connection and prove that it is skew-symmetric. When an affine connection is said to be symmetric? 3+2+2+2

8. *(a)* Define a Riemannian manifold. Prove that an Euclidean space is a Riemannian manifold. State Koszul's formula and the fundamental theorem of Riemannian geometry. 2+2+3+2

(b) Define an Einstein manifold. Prove that every 3-dimensional Einstein manifold is a manifold of constant curvature. 2+7

SECTION—C

Answer **all** questions.

9. Define a Nijenhuis tensor with respect to a vector-valued function ϕ. When an almost complex manifold is called a complex manifold? 1+2

10. Define curvature tensor and prove that it is skew-symmetric. 3

11. Prove that for all $\alpha, \beta \in \mathbb{R}$ and for all $X, Y, Z \in \chi(M)$

$$T(\alpha X + \beta Y, Z) = \alpha T(X, Z) + \beta T(Y, Z)$$ 3

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