PG-II (DE)/2014

2014

## MATHEMATICS

### Paper : VI

### (Unit - I)

Time - Two Hours Full Marks - 40

Answer *one* from **Section-A**, *three* from **Section-B** and *all* from **Section-C**.

*The figures in the margin indicate full marks.*

### Section - A

1. (a) Let $\lambda$ be a signed measure on a measurable space $(X, S)$. Let $A \in S$ be such that $\lambda(A) > 0$. Prove that there exists $B \in S$ such that $B$ is a positive set w.r. to $\lambda$, $B \subset A$ and $\lambda(B) > 0$. 9

(b) Define Lebesgue decomposition of a signed measure $\lambda$ w.r. to a positive measure $\mu$ on a measurable space $(X, S)$. Prove that if $\lambda$ is $\sigma$-finite and if $\{\lambda_1', \lambda_2'\}$, $\{\lambda_1'', \lambda_2''\}$ be two Lebesgue decompositions of $\lambda$ w.r. to $\mu$, then $\lambda_1' = \lambda_1''$, $\lambda_2' = \lambda_2''$ on $S$. 9

2. (a) Let $(X \times Y, \sigma(S \times T), \mu_1 \times \mu_2)$ be the product measure space of two $\sigma$-finite measure spaces $(X, S, \mu_1)$ and $(Y, T, \mu_2)$. Prove that for every $A \in \sigma(S \times T)$, the function $F(x) = \mu_2(A_x) : X \to \overline{\mathbb{R}}$ is $S$-measurable and

$$(\mu_1 \times \mu_2)(A) = \int_X F(x)\, d\mu. \qquad 9$$

(b) Let $(\mathbb{R}, M(\mu_g^*), \mu_g)$ be a Lebesgue-Stieltjes measure space on $\mathbb{R}$ determined by a real-valued increasing function $g$ on $\mathbb{R}$. Show that $(\mathbb{R}, M(\mu_g^*), \mu_g)$, is finite if and only if $g$ is bounded on $\mathbb{R}$. 9

## Section - B

3. "If $\{B_n\}$ is an increasing sequence of negative sets for a signed measure $\lambda$, then $\lim B_n$ is also a negative set for $\lambda$".—Prove or disprove. 6

4. Define :

(i) mutual singularity of two signed measures;

(ii) absolute continuity of a signed measure w.r. to a positive measure.

If $\lambda$ and $\mu$ are mutually singular and $\lambda$ is absolutely continuous w.r. to $\mu$, prove that $\lambda(A) = 0$ for all $A \in S$. 6

5. Let $\mu$ be a finite measure and $\{\gamma_n\}$ be a sequence of finite measures on a measurable space $(X, S)$ such that for all $n \in \mathbb{N}$, $\gamma_n << \mu$. Assume that $\lim \gamma_n(E)$ exists in $\overline{\mathbb{R}}$ for each $E \in S$. Show that the function $\nu : S \rightarrow [0, \infty]$ defined by $\nu(E) = \lim \gamma_n(E)$ is a measure such that $\nu << \mu$. 6

6. Let $(X \times Y, \sigma(S \times T), \mu \times \nu)$ be the product measure space of two $\sigma$-finite measure spaces $(X, S, \mu)$ and $(Y, T, \nu)$. Let $w$ be a non-negative extended real-valued $\sigma(S \times T)$ measurable function on $X \times Y$. Show that $F(x) = \int_X w^y \, d\mu$ is a $\Gamma$-measurable function of $y \in Y$ and

$$\int_{X \times Y} wd(\mu \times \nu) = \int_Y F \, d\nu. \qquad 6$$

7. Verify the following statements about signed measures :

(a) If $\mu << w$ and $\nu << \mu$, then $|\mu| + |\nu| << w$.

(b) If $\mu << w$ and $|\nu| + |\mu|$, then $\nu << w$. 6

## Section - C

8. Given two sequences of subsets $\{E_n\}$ and $\{F_n\}$ of a set $X$. Show that

$$\lim_{n \to \infty} \sup (E_n \cup F_n) \subset \left(\lim_{n \to \infty} \sup E_n\right) \cup \left(\lim_{n \to \infty} \sup F_n\right).$$

2

9. Show that the class of semi-algebras of subsets of a non-empty set may not be closed under finite unions. 2

2014

# MATHEMATICS

## Paper : VI

## (Unit - II)

Time - Two Hours Full Marks - 40

Answer *one* from **Section-A**, *three* from **Section-B** and *all* from **Section-C**.

*The figures in the margin indicate full marks.*

***Notations and symbols have their usual meanings.***

### Section - A

1. (a) Show that every bounded linear functional $f$ on $C[a, b]$ can be represented by a Riemann-Stieltjes integral $f(x) = \int_a^b x(t)dw(t)$ where $w$ is of bounded variation on $[a, b]$ and has the total variation $\text{Var}(w) = \|f\|$. 9

(b) Let $H_1, H_2$ be Hilbert spaces over the same field $K$ and $h : H_1 \times H_2 \to K$ a bounded sesquilinear form. Then show that $h$ has a representation $h(x, y) = \langle Sx, y \rangle$ where $S : H_1 \to H_2$ is a bounded linear operator uniquely determined by $h$ and has norm $\| S \| = \| h \|$. 9

2. (a) Show that every bounded linear operator $T$ from a Banach space $X$ onto a Banach space $Y$ is an open mapping. If $T$ is bijective then show that $T^{-1}$ is continuous. 10+2=12

(b) Find dual of $l_1$. 6

## Section - B

3. Show that there is no distinction between strong and weak convergence of sequences if the underlying normed linear space is of finite dimension. 6

4. Show that every Hilbert space is a Banach space but a Banach space is a Hilbert space only if parallelogram law holds therein. 6

5. Show that dual of an inner product space is a Hilbert space. 6

6. Show that equivalent norms yield same topologies on the underlying linear space. 6

7. Show that every Hilbert space is reflexive but it is not true for Banach spaces. 4+2=6

## Section - C

8. Check the continuity of the operator $T : l_2 \to l_2$ defined by

$$T(\xi_1, \xi_2, \cdots\cdots) = (0, \xi_2, \xi_3, \cdots\cdots). \qquad 2$$

9. Let $D[a, b]$ be the collection of all real-valued continuously differentiable functions on $[a, b]$. Then $D[a, b]$ is a subspace of the Banach space $C[a, b]$ with sup norm. Consider the operator $D : D[a, b] \to C[a, b]$ defined by $D(x) = \frac{dx(t)}{dt} \; \forall \, x \in D[a, b]$. Check the boundedness of $D$. 2

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2014

# MATHEMATICS

## Paper : VII

## (Unit– I)

Time - Two Hours Full Marks - 40

*The figures in the margin indicate full marks.*

***Symbols have their usual meanings.***

### Section - A

Answer ***any one*** question : 18×1=18

1. Consider the wave equation

$$u_{tt}(x,t)-\left(\frac{1}{2}\right)^2 u_{xx}(x,t)=0, -\infty<x<\infty, 0<t$$

with the initial data

$$u(x,o)=f(x)=\begin{cases}0, & x\le -1\\ 1-x^2, & -1<x\le 1,\\ 0, & 1<x.\end{cases} \quad u_t(x,o)=0,$$

(a) Find the solution. 10

(b) Sketch the profiles of solution at $t=1, 2, 3.$ 8

2. Show that the Dirichlet problem for Laplace's equation is stable and solve this problem for the upper half plane. 18

## Section - B

Answer ***any three*** questions : $6\times3=18$

3. Determine the integral surfaces of the equation

$$x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u,$$

with the initial data

$$x+y=0,\ u=1.$$ 6

4. Show that a solution of the Neumann problem

$$\nabla^2 u = f \text{ in } D$$
$$u = g \text{ on } B$$

differs from another solution by at most a constant. 6

5. Find the complete integral of the PDE :

$z^2 = pqxy$ using Charpits Method. 6

6. Show the following PDEs :

$$xp - yq = x \text{ and } x^2p + q = xz$$

are compatible and hence, find their solutions. 6

7. Solve the problem of vibration of a string of finite length by separation of variables method. 6

## Section - C

Answer ***all*** questions : $2\times2=4$

8. Classify the second order PDE :

$$(1-x^2)u_{xx} + u_{yy} = 0.$$ 2

9. Does the PDE :

$$u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0$$

admit characteristic curves ? Give reasons. 2

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**2014**

**MATHEMATICS**

**Paper : VII**

**(Unit - II)**

Time - Two Hours　　　Full Marks - 40

Answer ***one*** from **Section-A**, ***three*** from **Section-B** and ***all*** from **Section-C**.

*The figures in the margin indicate full marks.*

## Section - A

1. (a) Explain the derivation of the Hamilton-Jacobi equation. Discuss significance of Hamilton's characteristic function. 10+2=12

(b) State and prove Jacobi's theorem. 6

2. Distinguish between holonomic and non-holonomic constraint with one example of each. Describe the derivation of Lagrange's equations of motion of the first kind for $N$ particle system with $k$ constraints. 4+14=18

## Section - B

3. Derive the equation of motion of simple pendulum from Hamilton's equations of motion. 6

4. Define generating function of a canonical transformation. Show that $P = \frac{1}{2}(p^2 + q^2)$ and $Q = \tan^{-1}\frac{q}{p}$ is a canonical transformation. 6

5. Find the Lagrangian for a particle of mass $m$ moving under a central force. Find the corresponding equations of motion. 6

6. Determine the potential of uniform circular plate of finite radius. Find its value at the centre, 6

7. Determine the curve along which a particle falling from rest under the influence of gravity travels from higher to lower point in minimum time. 6

## Section - C

8. Explain :

   (i) generalized coordinate

   (ii) degrees of freedom. 2

9. Show that $[t, H] = 1$, $H = H(q, p, t)$ being the Hamiltonian. 2

2014

# MATHEMATICS

## Paper : VIII

## (Unit - I)

Time - Two Hours Full Marks - 40

Answer ***one*** from **Section-A**, ***three*** from **Section-B** and ***all*** from **Section-C**.

*The figures in the margin indicate full marks.*

## Section - A

1. (a) Derive Euler's equation of motion for perfect fluid. 6

(b) A flow field on the $xy$-plane has the velocity components

$$u = 3x + y,\ v = 2x - 3y.$$

Find the circulation around the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. 6

(c) Show that across the streamline there is no flow. 2

(d) Write short notes on : 2×2=4

(i) Vortex Doublet

(ii) Stokes' stream function.

2. (a) If a velocity field is given by $\overline{q} = u\bar{i} + v\bar{j} + w\bar{k}$ where $u = a(x^2 - y^2)$, $v = 2axy$, $w = 0$ then find stream function and velocity potential. 6

(b) Show that the image of a source with respect to a circle is a source of same strength at the inverse point and a sink of the same strength at the centre. 6

(c) Write short notes on : (i) Doublets, (ii) Irrotational flow, (iii) Non-Newtonian-fluid. 3×2=6

## Section - B

3. Show that single row of infinite vertices is unstable. 6

4. State Blasius theorem. Calculate the force on a circular cylinder due to a doublet outside it. 6

5. Give the definitions of the following non-dimensional coefficients : 3×2=6

(i) Lift and drag coefficients

(ii) Local skin friction coefficients

(iii) Nusselt number for heat transfer.

6. State and prove Kelvin's circulation theorem. 6

7. State and prove Helmholtz's vorticity theorem. 6

## Section - C

8. Give an example of a source and interpret it. 2

9. Define viscous fluid flow. 2

2014

# MATHEMATICS

## Paper : VIII

## (Unit - II)

Time - Two Hours Full Marks - 40

Answer ***one*** from **Section-A**, ***three*** from **Section-B** and ***all*** from **Section-C**.

*The figures in the margin indicate full marks.*

***Symbols have their usual meanings.***

### Section - A

1. (a) Establish Navier-Stokes equations of motion for a viscous compressible fluid in vector form. 10

(b) Derive Crocco's equation for steady isentropic flow. 4

(c) By introducing $\Psi$ in the Navier-Stokes equation, show that the order of the equation can be increased by two. 4

2. (a) Derive Prandtle's boundary layer equations. 6

(b) Establish the equation of energy of any element of a non-viscous non-conducting compressible fluid in the following form :

$$\frac{dE}{dt} = \frac{1}{\rho}\frac{dp}{dt}. \qquad 6$$

(c) Establish Rankine-Hugoniot relations for a normal shock wave. 6

**Section - B**

3. Determine the velocity profile when the flow is along the annular section bounded by two concentric circular cylinders. 6

4. Discuss the unsteady flow of viscous incompressible fluid over a suddenly accelerated flat plate. 6

5. Establish Prandtle's relation for an oblique shock wave.

6. Determine the displacement thickness and momentum thickness for the laminar boundary layer on a flat plate for which the velocity distribution is given by the relation

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4. \qquad 6$$

7. Find the pressure distribution such that the velocity field is given by

$$v_1 = k\,(x_1^2 - x_2^2),\ v_2 = -\,2k\,x_1\,x_2,\ v_3 = 0,$$

where $k$ is a constant, satisfies Navier-Stokes equation for an incompressible fluid in the absence of body force. 6

## Section - C

8. Prove that $C_p - C_v = R$. 2

9. What is the importance of Prandtle's boundary layer theory in fluid dynamics ? 2

2014

# MATHEMATICS

## Paper : IX

## (Unit - I)

Time - Two Hours                Full Marks - 40

*The figures in the margin indicate full marks.*

### Section - A

1. Explain the concept of the principle of equivalence and covariance in general relativity. 2+2=4

### Section - B

Answer ***any three*** questions : 3×6=18

2. Explain the concepts of proper time and distance in a gravitational field. Calculate them in the case of Schwarzschild exterior metric. 2+2+2=6

3. What do you mean by time dilation in special theory of relativity ? Show that the wave equation remains invariant under Lorentz transformation. 3+3=6

4. Show that the deflection of light just grazing the Sun is

$$\Delta\phi = 1.75''.$$ 6

5. Show thet Lorentz transformations form a group. 6

6. Write down Schwarzschild exterior solution in standard and isotropic coordinates. 3+3=6

## Section - C

Answer ***any one*** question : 1×18=18

7. Find the constants of motion in the Schwarzschild field of gravity. 18

8. Show that the advance of perihelion of Mercury is 43″/Earth Century. 18

**2014**

**MATHEMATICS**

**Paper : IX**

**(Unit - II)**

Time - Two Hours Full Marks - 40

*The figures in the margin indicate full marks.*

**Section - A**

Answer ***any one*** question : 1×18=18

1. State Cosmological Principle. Using it, derive the Robertson-Walker metric.

2. Assuming the dynamical equations of cosmology with scale factor $S(t)$ and for $k = +1$, show that the life span of the universe is

$$t_L = \frac{2\pi q_0}{(2q_0 - 1)^{3/2}} \cdot \frac{1}{H_0}$$

where the terms have their standard meanings.

## Section - B

Answer ***any three*** questions : $3 \times 6 = 18$

3. Deduce the dynamical equations of cosmology in the form

$$R^2 + k = \frac{8\pi}{3}\,\rho R^2$$

$$\rho + 3(p+\rho)\frac{R}{R} = 0,$$

where the terms have their standard meanings.

4. Define a "hot" universe and argue that the radiation maintains its black body spectrum throughout the early universe till the onset of the matter dominated era.

5. Discuss the concepts of particle and event horizon in cosmology.

6. Deduce that the scale factor $S(t)$ in the $\lambda$–cosmology with $k = +1$ satisfies the differential equation

$$S^2 = c^2\left(\frac{1}{3}\lambda S^2 - 1 + \frac{\gamma}{S}\right)$$

where

$$\gamma = \frac{2q_0 + \frac{2}{3}\frac{\lambda c^2}{H_0^2}}{\left[2q_0 - 1 + \frac{\lambda c^2}{H_0^2}\right]^{3/2}}\left(\frac{c}{H_0}\right),$$

the involved terms have standard meanings.

7. Write down the Brands-Dicke action and the field equations following from it. Discuss the cases when the integration constant $C = 0$ and $C \neq 0$.

## Section - C

Answer ***all*** the questions : 2×2=4

8. Deduce Hubble law in the form

$$cz = H_0 D_1 .$$

9. Starting from the baryon conservation law, show that the black body radiation spectrum is preserved during cosmological evolution.

PG-II (DE)/2014

2014

**MATHEMATICS**

**Paper : X**

**(Unit - I)**

Time - Two Hours | Full Marks - 40

*The figures in the margin indicate full marks.*

## Section - A

Discuss briefly :

1. Rydberg – Scuster law. 2

2. Albedo. 2

## Section - B

Answer ***any three*** questions.

3. Deduce second approximate forms for ***J***, ***H*** and ***K*** using Eddington's treatment. 6

4. What is impact broadening ? Discuss it briefly. 2+4=6

5. Find an expression for oscillator strength in emission line using transition coefficient. 6

6. Clearly explain the term "Interlocking of absorption lines". 6

7. Show that when $K_\nu$ is independent of $\nu$, $J = B(T)$. 6

## Section - C

Answer *any one* question.

8. Deduce the equation of transfer for the line formation when the ratio of line absorption co-efficient to the absorption co-efficient is under the influence of arbitrary variations and solve it. 18

9. Deduce the equation of transfer for interlocked doublets and solve it by Eddington's method. 18

2014

# MATHEMATICS

## Paper : X

## (Unit - II)

Time - Two Hours Full Marks - 40

*The figures in the margin indicate full marks.*

## Section - A

Discuss briefly :

1. $H$ – function. 2

2. Milne's integral equation. 2

## Section - B

Answer ***any three*** questions.

3. Deduce the relation, $b(s) = B(0)\, B(0, s)$. 6

4. Show that, $h_0 = 1 \pm \sqrt{(1 - 2\psi_0)}$. 6

5. Solve Milne's integral equation by Ambertzumian's method. 6

6. Write down the recurrence relation satisfied by the Legendre's polynomials used in the double interval SHM. How the phase function is represented using Legendre polynomials in case of anisotropic scattering ? 2+4=6

7. Show that, $L_\tau\{1\} = 2\psi_0 - K_2(\tau)$. 6

## Section - C

Answer ***any one*** question.

8. Solve the equation of transfer in a semi-infinite, isotropically scattering atmosphere by spherically harmonic method (SHM). 18

9. Prove that functions $X(\mu)$ and $Y(\mu)$ satisfy the following equations,

$$X(\mu) = 1 + \mu \int_0^1 \left\{ \frac{X(\mu)\, X(u) - Y(\mu)\, Y(u)}{\mu + u} \right\} \psi(u)\, du$$

$$Y(\mu) = \exp\left(-\tau_1/\mu\right) + \mu \int_0^1 \left\{ \frac{Y(\mu)\, X(u) - X(\mu)\, Y(u)}{\mu - u} \right\} \psi(u)\, du$$

where the symbols have their usual meaning. 18